‘Your first calculus book’

Calculus in ten easy pieces

“THE NEW SUPER EASY SELF – START BOOK FOR BEGINNERS”

FOREWORD

In a way, physics is applied mathematics. So, while teaching physics a lot of mathematics is assumed to be known. But this may not be the case in general. One such glaring example is that of concepts in calculus. Students not only of physics but also of commerce and economics, are expected to have good knowledge of differentiation and integration, which they might not have studied in their regular curriculum.

This small book is prepared mainly to help students facing such difficulties. The topics considered in this book are extremely important and they are explained using examples which are very easy to understand. No where the proofs are included. More importance is given to learn how to apply the formula, once it is known.

I wish the readers of this book, a better understanding of how to do differentiation and integration that are needed in this subject. I also wish a great success to the enthusiastic author.

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INTRODUCTION

This book is written from the perspective of a beginner who wants to learn some basic techniques of differentiation and integration in an ultrashort span of time. The book doesn't dig into the history, nature and fundamental principles of calculus which if required, the students will anyway learn in their formal mathematics classes. This is a simple and student friendly book which is intended to give you the cheerful mood of a coffee table book, but at the same time giving you a very essential, clear and uncluttered entry to begin your study of calculus. In simple words this book teaches you how to 'do calculus' rather than discuss the philosophy and underlying principles of calculus.

The language used in the book is extremely simple and informal. Each step in every concept or problem is explained with plenty of minor details. This can be used as a self-study book by any grade 9 or 10 student to get a head start. Alternatively, this book can be used by pre-university (grade 11) teachers to conduct a 3-days workshop in calculus for students, before starting any standard physics course or any other related subject course, where calculus is a pre requisite. To sum it up, I am sure, anyone who is interested to learn the basics of differentiation and integration will find the easy approach used in the book very appealing and enjoyable.

The Author

TABLE OF TEN EASY PIECES

DIFFERENTIATION

Chapter - 1 ALGEBRAIC FUNCTIONS

Let us have an algebraic function of the form $\boldsymbol{y = ax^b}$, here '$\boldsymbol{a}$' is a constant called as co-efficient and '$\boldsymbol{b}$' is known as an exponent and '$\boldsymbol{x}$' is called the variable. Now, we are going to learn to perform a mathematical operation called as differentiation on the above function, it is also known as finding the derivative of the function.

Note: Differentiation is always done with respect to a variable. Here in our function $\boldsymbol{y = ax^b}$ the variable is $\boldsymbol{x}$, so we say, now we are going to learn how to differentiate the function $\boldsymbol{y = ax^b}$ with respect to $\boldsymbol{x}$.

Let us see how to do it.

To differentiate the function with respect to $\boldsymbol{x}$, or in other words to find the derivative of this function with respect to $\boldsymbol{x}$, you must carry out a fairly simple task as explained below.

"First multiply the constant co-efficient with the exponent, and then subtract the value of the exponent by 1".

Look at the formula given below,

Formula: $\boldsymbol{\frac{d}{dx}(ax^b) = b\,.\,a\,.\,x^{b-1}}$

Let us explore the formula a bit more carefully.

First see the L.H.S, $\boldsymbol{\frac{d}{dx}(ax^b)}$ is read as differentiation of $(\boldsymbol{ax^b})$ with respect to $\boldsymbol{x}$.

Now, see the R.H.S, $\boldsymbol{b\,.\,a\,.\,x^{b-1}}$, as said earlier, we have first multiplied the constant co-efficient $\boldsymbol{a}$ with the exponent $\boldsymbol{b}$. And then we have reduced the value of exponent by 1 $(\boldsymbol{i.e.\,b-1})$. This is all that you have to do.

Let us make everything clear, by taking an example.

Example 1: Say, you are asked to find the derivative of a function $y = 5x^4$ (w.r.t x – HENCEFORTH IN THIS BOOK , WITH RESPECT TO WILL BE WRITTEN AS w.r.t)

How do we do it? Simple enough.

First compare the given equation $5x^4$ with ax^b.

You can clearly say that, by comparison $a = 5$ and $b = 4$.

Now let us apply the formula we learnt, and insert the values correctly.

We know the formula for differentiation,

$\frac{d}{dx}(ax^b) = b.\ a\ .\ x^{b-1}$

Therefore, derivative of $5x^4$ is written as $\frac{d}{dx}(5x^4)$.

Inserting $a = 5$ and $b = 4$ in the R.H.S of the above formula,

We get, $= 4 \times 5\ \times x^{4-1} = 20\ \times x^{4-1} = 20 \times\ x^3$

That's all, you have successfully differentiated the function $y = 5\,x^4$ w.r.t x and the answer is $20\,x^3$.

Example 2: Find the derivative of $y = 3x^9$ w.r.t x.

First compare $y = 3\,x^9$ with $y = a\,x^b$, We can see that $a = 3$ and $b = 9$. Now, insert the right values in the formula $\frac{d}{dx}(ax^b) = b.\ a.\ x^{b-1}$

We get, $\frac{d}{dx}(3\,x^9) = 9 \times 3\ \times x^{9-1} = 27\,x^{9-1}$

$= 27\ x^8$, is our answer.

Example 3: Find the derivative $\frac{d}{du}(u^8)$, with respect to u.

What do you see here? Simple, instead of usual x used in mathematics, the variable used above is u. Just replace x by u in the formula and carry out the same process.

So now the formula $\frac{d}{dx}(a\,x^b) = b.a\,.\,x^{b-1}$ becomes $\frac{d}{du}(a\,u^b) = b\,.\,a\,.\,u^{b-1}$

let us take the given function u^8 and compare it with $a\,u^b$. You may easily say $b = 8$, but what is a equal to? The answer is $a = 1$, why? Because, u^8 is same as $1 \times u^8$. Hence $a = 1$ and $b = 8$.

Note: If no coefficient or exponent is seen then the number is taken as 1.

Therefore, $\frac{d}{du}(u^8) = \frac{d}{du}(1 \times u^8) = 8 \times 1 \times u^{8-1} = 8u^7$

Now you have learnt the basics of carrying out differentiation of algebraic functions successfully. Take your time to relearn and soon you will begin enjoying the process of finding derivatives of functions.

Let us move ahead. Sometimes the algebraic functions or polynomials look very interesting and different. Here are a few tips to bring it in correct form.

1. If you see a function in the form $y = \frac{5}{x^4}$, You may rewrite it as $y = 5x^{-4}$. Now you see, you can easily compare it with the form $y = a\,x^b$, which gives $a = 5$ and $b = -4$.

So, the hint here is, if you see the variable in the denominator, bring it to the numerator by changing the sign of the exponent Example: $\frac{1}{x^{10}} = x^{-10}$, $\frac{1}{x^b} = x^{-b}$ etc.

2. If you are asked to differentiate the function $y = 7\sqrt{x}$, first rewrite the function as $y = 7x^{1/2}$, then $a = 7$ and $b = \frac{1}{2}$, the hint here is if you see a square root, write it using an exponent $\frac{1}{2}$, similarly cube root can be written as exponent $\frac{1}{3}$ and so on.

Note the list of useful formula's used in exponents.	
$x^p x^q = x^{p+q}$,	$\frac{x^p}{x^q} = x^{p-q}$
$(x^p)^q = x^{pq}$	$(ax)^p = a^p x^p$
$\sqrt{ax} = a^{1/2}x^{1/2}$	$x^0 = 1$,

3. If you are asked to differentiate the number 10, what does that mean? And what is the answer? Here, there is no variable at all. In other words, 10 can also be written as $\mathbf{10x^0}$, compare with $\boldsymbol{a\,x^b}$.

We get $\boldsymbol{a = 10}$, $\boldsymbol{b = 0}$

So, the formula gives the R.H.S = $\boldsymbol{b\,.\ a\,.x^{b-1}}$

$$= \mathbf{0 \times 10 \times}\ \boldsymbol{x^{0-1}}$$

$$= \mathbf{0}$$

Since zero is multiplied, the final answer is zero. Therefore, the conclusion is "derivative of a constant" is always zero. $\boldsymbol{\frac{d}{dx}(Constant) = 0}$

Example : $\boldsymbol{\frac{d}{dx}(5) = 0}$, $\boldsymbol{\frac{d}{dx}(-75) = 0}$, $\boldsymbol{\frac{d}{dx}(100 + 125^3) = 0}$

All the functions are constants in the above examples.

4. If you are asked to differentiate $\boldsymbol{\frac{d}{dx}(9x^2 - 14x + 3)}$, then at a first glance you can see that function or polynomial directly is not of the form $\boldsymbol{a\,x^b}$. In such cases what you can do is, differentiate each term individually and sum up the final answers. Meaning,
write $\boldsymbol{\frac{d}{dx}(9x^2 - 14x + 3)}$ as $\boldsymbol{\frac{d}{dx}(9x^2) + \frac{d}{dx}(-14x) + \frac{d}{dx}(3)}$

Now you can do the differentiation of each term individually, as each term is separately in the form $\boldsymbol{a\,x^b}$.

$$\boldsymbol{\frac{d}{dx}(9x^2) = 9 \times 2 \times\ x^{2-1} = 18\,x^1}$$

$$\boldsymbol{\frac{d}{dx}(-14x) = (-14)(1)\,x^{1-1} = -14\,x^0 = -14}$$

$$\boldsymbol{\frac{d}{dx}(3) = 0\ \because Constant}$$

∴ The final answer is $\boldsymbol{(18x - 14 + 0) = 18x - 14}$

"If the polynomial has many terms, find the derivative of each term individually and then add them together"

$$\boldsymbol{\frac{d}{dx}(y_1 + y_2 + y_3 + \ \ldots\ldots\ldots\ldots\ldots + y_n) = \frac{d}{dx}(y_1) + \frac{d}{dx}(y_2) + \frac{d}{dx}(y_3) + \ \ldots\ldots\ldots\ldots + \frac{d}{dx}(y_n)}$$

Let us take another example for clear understanding.

Differentiate: $\frac{d}{dx}(\mathbf{10x^5 + 14x^2 + 12x + 9})$

Now compare with $\frac{d}{dx}(\mathbf{y_1 + y_2 + y_3 + \dots\dots\dots\dots\dots + y_n})$ then, $\mathbf{y_1 = 10x^5}$, $\mathbf{y_2 = 14x^2}$, $\mathbf{y_3 = 12x}$, and $\mathbf{y_4 = 9}$

$$\frac{d}{dx}(y_1) = \frac{d}{dx}(10x^5) = 5 \times 10 \times x^{5-1} = 50x^4$$

$$\frac{d}{dx}(y_2) = \frac{d}{dx}(14x^2) = 2 \times 14 \times x^{2-1} = 28x$$

$$\frac{d}{dx}(y_3) = \frac{d}{dx}(12x) = 1 \times 12 \times x^{1-1} = 12 \times x^0 = 12$$

$\frac{d}{dx}(\mathbf{y_4}) = \frac{d}{dx}(\mathbf{9}) = \mathbf{0}$ ∵ Constant

∴ The final answer is,

$$\frac{d}{dx}\left(10x^5 + 14x^2 + 12x + 9\right) = 50x^4 + 28x + 12 + 0$$

$$= 50x^4 + 28x + 12$$

As of now, this is all you need to remember while differentiating an algebraic function or a polynomial. Let us make our foundation and practice very strong by solving a few more exercise questions.

Exercises

Find the derivative of each of the following functions with respect to the given variable. Solve them by yourself first, and then only check the solutions to confirm your answers.

Exercise 1: $\frac{d}{dx}(\mathbf{3x^5})$

Solution: Compare with $\frac{d}{dx}(\mathbf{ax^b})$ and solve as per the formula $\frac{d}{dx}\left(\mathbf{ax^b}\right) = \mathbf{b\,.\,a\,.\,x^{b-1}}$, $\mathbf{3x^5}$ compared with $\mathbf{ax^b}$ gives $\mathbf{a = 3}$, $\mathbf{b = 5}$

Applying in the formula gives $= \mathbf{5 \times 3 \times x^{5-1} = 15 \times x^{5-1} = 15\,x^4}$

$$\therefore \frac{d}{dx}\left(3x^5\right) = 15\,x^4$$

Exercise 2: $\frac{d}{dt}\left(\frac{\mathbf{10}}{\mathbf{t}}\right)$

Solution: Here the variable is $\boldsymbol{t}$. So, we have to use the formula $\frac{d}{dt}\left(\boldsymbol{at^b}\right) = \boldsymbol{b\,.\ a\,.\ t^{b-1}}$.

Compare, $\frac{\mathbf{10}}{\boldsymbol{t}}$ with $\boldsymbol{at^b}$. As explained earlier, when variable is in the denominator, rewrite it in the numerator. $\frac{\mathbf{10}}{\boldsymbol{t}}$ can be rewritten as $\mathbf{10}\boldsymbol{t}^{-1}$, now comparison with $\boldsymbol{at^b}$ tells us that $\boldsymbol{a} = \mathbf{10}, \boldsymbol{b} = -\mathbf{1}$

Insert the values in the formula,

$$= (-1) \times (10) \times t^{-1-1} = -10 \times t^{-2} = -\frac{10}{t^2}$$

Exercise 3: $\frac{d}{dx}\left(\frac{5}{x^2}\right)$

Solution: First rewrite as $\frac{d}{dx}(5x^{-2})$, Comparing with $\boldsymbol{ax^b}$, $\boldsymbol{a} = \mathbf{5}$ and $\boldsymbol{b} = -\mathbf{2}$.
Applying the formula $\frac{d}{dx}\left(\boldsymbol{ax^b}\right) = \boldsymbol{b\,.\ a\,.\ x^{b-1}}$ and inserting right values,

$$\frac{d}{dx}(5x^{-2}) = (-2) \times (5) \times x^{-2-1} = -10 \times x^{-3} = -\frac{10}{x^3}$$

Exercise 4: $\frac{d}{dx}\left[9x^{(5/3)}\right]$

Solution: Compare $\mathbf{9}\boldsymbol{x}^{5/3}$ with $\boldsymbol{ax^b}$, $\boldsymbol{a} = \mathbf{9}$, $\boldsymbol{b} = \frac{5}{3}$

Applying the formula and inserting the values,

$$\frac{d}{dx}\left(9x^{5/3}\right) = \left(\frac{5}{3}\right) \times 9 \times x^{\left(\frac{5}{3}\right)-1} = 15 \times x^{\left(\frac{5}{3}\right)-1} = 15 \times x^{\left(\frac{5-3}{3}\right)}$$

$$= 15x^{\left(\frac{2}{3}\right)}$$

Exercise 5: $\frac{d}{dx}\left[\frac{x^{(10/3)}}{24}\right]$

Solution: Compare $\frac{x^{\left(\frac{10}{3}\right)}}{24}$ with $\boldsymbol{ax^b}$, $\boldsymbol{a} = \frac{1}{24}$ and $\boldsymbol{b} = \frac{10}{3}$

Now inserting these values in the formula, we get

$$\frac{d}{dx}\left[\frac{1}{24} \times x^{\left(\frac{10}{3}\right)}\right] = \frac{10}{3} \times \frac{1}{24} \times x^{\left(\frac{10}{3}-1\right)}$$

$$= \frac{10}{72} \times \; x^{\left(\frac{10-3}{3}\right)} = \frac{10}{72} \times \; x^{\left(\frac{7}{3}\right)}$$

$$= \frac{10x^{\left(\frac{7}{3}\right)}}{72}$$

$$= \frac{5x^{\left(\frac{7}{3}\right)}}{36}$$

Exercise 6: $\frac{d}{dt}(\sqrt{10t})$

Solution: First we must note that the variable is t. Therefore, the formula to be used here is $\frac{d}{dt}(at^b) = b\,.\,a\,.\,t^{b-1}$

Now to compare $\sqrt{10t}$ with at^b, first write $\sqrt{10t}$ as $(10t)^{\left(\frac{1}{2}\right)} = 10^{\left(\frac{1}{2}\right)} \times t^{\left(\frac{1}{2}\right)}$

∴ $10^{\left(\frac{1}{2}\right)} \times t^{\left(\frac{1}{2}\right)}$ when compared with at^b, we get $a = 10^{\left(\frac{1}{2}\right)}$ or $\sqrt{10}$ and $b = \frac{1}{2}$. Applying the formula and inserting the values we get,

$$\frac{d}{dt}(\sqrt{10t}) = \frac{d}{dt}\left(10^{\left(\frac{1}{2}\right)} \times t^{\left(\frac{1}{2}\right)}\right) = \frac{1}{2} \times 10^{\left(\frac{1}{2}\right)} \times t^{\left(\frac{1}{2}-1\right)}$$

$$= \frac{\sqrt{10}}{2} \times t^{\left(-\frac{1}{2}\right)}$$

Exercise 7: $\frac{d}{dx}(6x^3 - 4x^2 + 2x + 5)$

Solution: The function has multiple terms, so we have to individually differentiate each term and add the answers at the end.

$$\frac{d}{dx}(y_1 + y_2 + y_3 + \cdots + y_n) = \frac{d}{dx}(y_1) + \frac{d}{dx}(y_2) + \frac{d}{dx}(y_3) + \; \ldots \ldots + \; \frac{d}{dx}(y_n)$$

In the above question, $\frac{d}{dx}(6x^3 - 4x^2 + 2x + 5)$ can be written as

$$\frac{d}{dx}(6x^3) + \frac{d}{dx}(-4x^2) + \frac{d}{dx}(2x) + \; \frac{d}{dx}(5)$$

Now, we find the derivative individually,

$$\frac{d}{dx}(6x^3) = 3 \times 6 \times x^{3-1} = 18 \times x^{3-1} = 18\,x^2$$

$$\frac{d}{dx}(-4x^2) = 2 \times (-4) \times x^{2-1} = -8 \times x^1 = -8x$$

$$\frac{d}{dx}(2x) = \frac{d}{dx}(2x^1) = 1 \times 2 \times x^{1-1} = 2x^0 = 2$$

$$\frac{d}{dx}(5) = 0 \ \because 5 \ is \ a \ constant$$

The final answer is

$$= 18x^2 - 8x + 2 + 0$$

$$= 18x^2 - 8x + 2$$

Exercise 8: $\frac{d}{dx}(4\sqrt[3]{x} + 12\sqrt{x} + 3x^{\left(\frac{5}{3}\right)} + 2)$

Solutions:

$$\frac{d}{dx}(4\sqrt[3]{x} + 12\sqrt{x} + 3x^{\left(\frac{5}{3}\right)} + 2) = \frac{d}{dx}\left(4\sqrt[3]{x}\right) + \frac{d}{dx}\left(12\sqrt{x}\right) + \frac{d}{dx}(3x^{\left(\frac{5}{3}\right)}) + \frac{d}{dx}(2)$$

Let us solve one term at a time, $\frac{d}{dx}(4\sqrt[3]{x})$ rewrite $4\sqrt[3]{x}$ as $4x^{\left(\frac{1}{3}\right)}$

Compare $4x^{\left(\frac{1}{3}\right)}$with ax^b, $a = 4$, $b = \frac{1}{3}$

$$\frac{d}{dx}\left[4x^{\left(\frac{1}{3}\right)}\right] = \frac{1}{3} \times 4 \times x^{\left(\frac{1}{3}-1\right)} = \frac{1}{3} \times 4 \times x^{\left(-\frac{2}{3}\right)} = \frac{4}{3} \times \ x^{\left(-\frac{2}{3}\right)}$$

Similarly, $\frac{d}{dx}\left(12\sqrt{x}\right) = \frac{d}{dx}(12 \times x^{\left(\frac{1}{2}\right)} = \frac{1}{2} \times 12 \times x^{\left(\frac{1}{2}-1\right)} = 6 \times x^{\left(-\frac{1}{2}\right)}$

And, $\frac{d}{dx}\left[3x^{\left(\frac{5}{3}\right)}\right] = \frac{5}{3} \times 3 \times x^{\left(\frac{5}{3}-1\right)} = 5 \times x^{\left(\frac{5-3}{3}\right)} = 5 \times x^{\left(\frac{2}{3}\right)} = 5x^{\left(\frac{2}{3}\right)}$

$$\frac{d}{dx}(2) = 0$$

The final answer is

$$= \frac{4}{3} \times x^{\left(-\frac{2}{3}\right)} + 6 \times x^{\left(-\frac{1}{2}\right)} + 5 \times x^{\left(\frac{2}{3}\right)} + 0$$

$$= \frac{4}{3x^{\left(\frac{2}{3}\right)}} + \frac{6}{x^{\left(\frac{1}{2}\right)}} + 5x^{\left(\frac{2}{3}\right)}$$

Note:

1. The equation $y = a\,x^b$ can also be written as $f(x) = ax^b$, $f(x)$ is read as 'f of x' or 'function of x' , indicating that ax^b is a function where x is the variable.

Example: $f(x) = 3x^2$, $f(x) = 4x^3$, $f(x) = 10\sqrt{x} + 2x$

2. The derivative of the function $\frac{d}{dx}(y)$ or $\frac{d}{dx}(ax^b)$ can also be written as $f'(x)$ read as 'f prime x' or 'f dash of x'.

If $f(x) = ax^b$then $f'(x) = b\,.\,a\,.\,x^{b-1}$

$f'(x)$ is just another symbol for derivative of the function w.r.t x.